The $\omega \to \infty$ limit of Brans–Dicke theory

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Abstract

The standard tenet that Brans–Dicke theory reduces to general relativity in the $\omega \to \infty$ limit has been shown to be false when the trace of the matter energy–momentum tensor vanishes. The issue is clarified in a new approach and the asymptotic behaviour of the Brans–Dicke scalar is rigorously derived.

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1 Introduction

There is a surge of interest among theoretical physicists in Brans–Dicke (BD) [1] and scalar–tensor theories, motivated by the fact that the association of scalar fields to the metric tensor seems unavoidable in superstring theories [2]. In addition, the scalar–tensor theories of which BD theory is the prototype exhibit a conformal invariance property that mimics the conformal invariance of string theories at high energies [3]–[7], and is applied below. Additional interest in BD and scalar–tensor theories comes from the extended [8] and hyperextended [9] inflationary scenarios of the early universe. In spite of the fact that BD theory is the oldest and best known alternative to general relativity (GR), its essential features are not well understood. The standard tenet (see e.g. [10]) that GR is obtained in the $\omega \rightarrow \infty$ limit of BD theory has been shown to be false for many exact solutions [11]–[16]. In addition, while it is believed that the BD field ϕ has the asymptotic behaviour

$$\phi = \phi_0 + \mathrm{O}\left(\frac{1}{\omega}\right) \tag{1.1}$$

(where ϕ_0 is a constant) as $\omega \rightarrow \infty$ [10], for the above–mentioned solutions, one has instead the asymptotic behaviour [11]–[16]

$$\phi = \phi_0 + \mathrm{O}\left(\frac{1}{\sqrt{\omega}}\right) \; . \tag{1.2}$$

Only very recently it was realized that the anomaly in the $\omega \rightarrow \infty$ limit is associated to the vanishing of the trace $T = {T^\mu}_\mu$ of the matter energy–momentum tensor $T_{\mu\nu}$ [16]. This is a key point in the understanding of the $\omega \rightarrow \infty$ limit; the condition $T = 0$ signals conformal invariance, and it is natural to relate it to the conformal invariance property of the gravitational part of the BD action.

When $T = 0$, the entire BD action is invariant under a 1–parameter group of conformal transformations $\mathcal{F}_\alpha$, and a change $\omega \rightarrow \tilde{\omega}$ in the BD parameter is equivalent to a transformation $\mathcal{F}_\alpha$ which moves a BD theory within an equivalence class $\mathcal{E}$. The $\omega \rightarrow \infty$ limit can also be seen as a parameter change that moves BD theory within the same class $\mathcal{E}$, and therefore it cannot reproduce GR, which does not belong to $\mathcal{E}$. On the other hand, when $T \neq 0$, the conformal invariance of BD theory is broken, the parameter change $\omega \rightarrow \tilde{\omega}$ and the $\omega \rightarrow \infty$ limit cannot be seen as a conformal transformation. One does not move within a equivalence class which excludes GR, and the $\omega \rightarrow \infty$ limit can then reproduce GR.

Finally, the asymptotic behaviour (1.2) of the BD scalar ϕ was obtained as a order

of magnitude estimate [16]; using the conformal transformation approach, Eq. (1.2) can be rigorously derived.

Our notations and conventions are as follows: the metric signature is $-+++$, the Riemann tensor is given in terms of the Christoffel symbols by $R_{\mu\nu\rho}{}^{\sigma} = \Gamma^{\sigma}_{\mu\rho,\nu} - \Gamma^{\sigma}_{\nu\rho,\mu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\sigma}_{\alpha\mu}$, the Ricci tensor is $R_{\mu\rho} \equiv R_{\mu\nu\rho}{}^{\nu}$, and $R = g^{\alpha\beta}R_{\alpha\beta}$. ∇_{μ} is the covariant derivative operator, $\Box \equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$, and we use units in which the speed of light and Newton constant assume the value unity.

2 Conformal invariance

We begin by considering the BD action in the Jordan conformal frame

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R + \frac{\omega}{\phi}\, g^{\alpha\beta}\nabla_{\alpha}\phi\nabla_{\beta}\phi\right] + S_{matter} \ , \tag{2.1}$$

where S_{matter} is the nongravitational part of the action, which is independent of ϕ. The field equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}\, T_{\mu\nu} + \frac{\omega}{\phi^2}\left(\nabla_{\mu}\phi\nabla_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\nabla^{\alpha}\phi\nabla_{\alpha}\phi\right) + \frac{1}{\phi}\left(\nabla_{\mu}\nabla_{\nu}\phi - g_{\mu\nu}\Box\phi\right) \ , \tag{2.2}$$

$$\Box\phi = \frac{8\pi T}{3+2\omega} \ . \tag{2.3}$$

Let us restrict, for the moment, to consider the purely gravitational sector of the theory: under the conformal transformation

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \ , \tag{2.4}$$

where $\Omega(x^{\alpha})$ is a nonvanishing smooth function, the Ricci curvature and the Jacobian determinant $\sqrt{-g}$ transform as [17]

$$\tilde{R} = \Omega^{-2}\left[R + \frac{6\Box\Omega}{\Omega}\right] \qquad , \qquad \sqrt{-\tilde{g}} = \Omega^4\sqrt{-g} \ , \tag{2.5}$$

and the BD Lagrangian density can be rewritten as follows

$$\mathcal{L}_{BD}\sqrt{-g} = \sqrt{-\tilde{g}}\left[\Omega^{-2}\phi\tilde{R} - \frac{6\phi\Box\Omega}{\Omega^5} + \frac{\omega}{\Omega^2\phi}\,\tilde{g}^{\mu\nu}\tilde{\nabla}_{\mu}\phi\tilde{\nabla}_{\nu}\phi\right] \ . \tag{2.6}$$

By specifying the conformal factor as

$$\Omega = \phi^{\alpha} \tag{2.7}$$

($\alpha \neq 1/2$) and by redefining the scalar field according to

$$\phi \longrightarrow \sigma = \phi^{1-2\alpha} \ , \tag{2.8}$$

one obtains

$$\mathcal{L}_{BD}\sqrt{-g} = \sqrt{-\tilde{g}}\left[\sigma\tilde{R} + \frac{\tilde{\omega}}{\sigma}\,\tilde{g}^{\mu\nu}\tilde{\nabla}_{\mu}\sigma\tilde{\nabla}_{\nu}\sigma\right] \ , \tag{2.9}$$

where

$$\tilde{\omega} = \frac{\omega - 6\alpha\left(\alpha - 1\right)}{\left(1-2\alpha\right)^{2}} \ . \tag{2.10}$$

Hence, the gravitational part of the BD action is invariant in form under the transformation given by Eqs. (2.4), (2.7), and (2.8). These transformations constitute a one–parameter Abelian group with a singularity in the parameter dependence at $\alpha = 1/2$. In fact, the consecutive action of two maps $\mathcal{F}_{\alpha}$, $\mathcal{F}_{\beta}$ of the kind (2.4), (2.7), (2.8) with parameters α and β gives a transformation $\mathcal{F}_{\gamma}$ of the same kind with parameter $\gamma\left(\alpha,\beta\right) = \alpha + \beta - 2\alpha\beta$, and $\alpha, \beta \neq 1/2$ implies $\gamma \neq 1/2$. The identity corresponds to the transformation $\mathcal{F}_0$ for $\alpha < 1/2$. The inverse of the transformation $\mathcal{F}_{\alpha}$ is the map $\mathcal{F}_{\bar{\alpha}}$ with $\bar{\alpha} = -\alpha\left(1-2\alpha\right)^{-1}$ for $\alpha < 1/2$. Finally, since $\gamma\left(\alpha,\beta\right) = \gamma\left(\beta,\alpha\right)$, the group is commutative.

If M is a 4–dimensional smooth manifold, the BD spacetimes $\left(M, g_{\mu\nu}^{(\omega)}, \phi^{(\omega)}\right)$ related by a transformation $\mathcal{F}_{\alpha}$ constitute an equivalence class $\mathcal{E}$.

If one adds ordinary (i.e. other than the BD scalar) matter to the BD action the conformal invariance is, in general, broken. However, under the conditions $T_{\mu\nu} = T_{\nu\mu}$ and $T = 0$ for the matter stress–energy tensor, the conservation equation

$$\nabla^{\nu}T_{\mu\nu} = 0 \tag{2.11}$$

(which contains the dynamics of matter) is conformally invariant [17]. Since $T_{\mu\nu}$ is not affected by the field redefinition (2.8), the total BD action is invariant under the group of transformations (2.4), (2.7), (2.8) if $T = 0$. In this case, a change of the BD parameter $\omega \rightarrow \tilde{\omega}$ is equivalent to a transformation $\mathcal{F}_{\alpha}$ of the kind (2.4), (2.7), (2.8) for a suitable value of the parameter $\alpha \neq 1/2$. Such a transformation $\mathcal{F}_{\alpha}$ maps the BD spacetime $(M, g_{\mu\nu}, \phi)$ corresponding to the value ω of the parameter into another spacetime $(M, \tilde{g}_{\mu\nu}, \sigma)$ corresponding to the value $\tilde{\omega}$ of the BD parameter, which belongs

to $\mathcal{E}$, and so does the $\omega \to \infty$ limit. Hence, by performing this limit, one cannot obtain GR solutions, because the latters do not belong to the equivalence class $\mathcal{E}$ (GR is invariant under diffeomorphisms, but not under conformal transformations). When matter with $T \neq 0$ is added to the BD Lagrangian, the conformal equivalence is broken, one no longer moves within the equivalence class $\mathcal{E}$ in the $\omega \to \infty$ limit, and it is possible to obtain GR.

Let us consider the singularity $\alpha = 1/2$ in the function $\tilde{\omega}(\alpha)$ given by Eq. (2.10); we restrict the discussion to the range of values $\omega > -3/2$ (the case $\omega < -3/2$ is symmetric). $\tilde{\omega}(\alpha)$ is singular at $\alpha = 1/2$, and has two branches; $\tilde{\omega} = \omega$ at $\alpha = 0$ and $\alpha = 1$, which correspond to the identity $\mathcal{F}_0$ in the group of transformations (2.4), (2.7), (2.8). The $\alpha \to 1/2$ limit corresponds to the $\tilde{\omega} \to \infty$ limit of the BD parameter; when $\alpha = 1/2$, the conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \phi\, g_{\mu\nu} \tag{2.12}$$

and the scalar field redefinition

$$\tilde{\phi} = \int \frac{(3+2\omega)^{1/2}}{\phi}\, d\phi \tag{2.13}$$

(instead of Eq. (2.8), which becomes meaningless), recast the theory in the so–called Einstein conformal frame (also called “Pauli frame” in Refs. [3, 4, 18]), in which the gravitational part of the action becomes that of Einstein gravity plus a non self–interacting scalar field as a material source

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\, \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} \right] . \tag{2.14}$$

The transformation (2.12), (2.13) has been known since the original BD paper [1] and it has later been generalized to scalar–tensor and nonlinear gravity theories, and rediscovered a number of times (see references in [19]). The BD parameter disappears, and the $\omega \to \infty$ limit cannot be considered: the theory is already GR, apart from a violation of the equivalence principle due to the anomalous coupling of the scalar to the energy–momentum tensor of ordinary matter, if $T_{\mu\nu} \neq 0$ ([19] and references therein).

Formally, BD theory with $\omega = -3/2$, which corresponds to the $\alpha \to \pm\infty$ limit, is a fixed point of the transformation (2.4), (2.7), (2.8); in fact Eq. (2.10) gives $\tilde{\omega} = \omega = -3/2$. However, the BD field equations are not defined in this case.

3 Asymptotic behaviour of the Brans–Dicke scalar as $\omega \rightarrow \infty$

The asymptotic behaviour (1.2) of the BD scalar in the $\omega \rightarrow \infty$ limit was derived in [16] as a order of magnitude estimate; using the conformal transformation approach, it is straightforward to provide a rigorous mathematical derivation of Eq. (1.2).

Under the condition $T = 0$, any value of the BD parameter $\tilde{\omega}$ can be obtained starting from a fixed value ω (cf. Eq. (2.10)). Without loss of generality, we start from the value $\omega = 0$ and solve Eq. (2.10) with respect to α to obtain

$$\alpha = \frac{1}{2}\left(1 \pm \frac{\sqrt{3}}{\sqrt{3+2\tilde{\omega}}}\right) \ . \tag{3.1}$$

When $\tilde{\omega} \rightarrow \infty$, $\alpha \rightarrow 1/2$ and Eq. (2.8) yields

$$\sigma \approx 1 \mp \left(\frac{3}{2\tilde{\omega}}\right)^{1/2} \ln \phi \tag{3.2}$$

as $\tilde{\omega} \rightarrow \infty$. The "old" BD field ϕ corresponding to $\omega = 0$ does not change in the limit, and the "new" BD field σ has the asymptotic behaviour given by Eq. (1.2).

The asymptotic behaviour (3.2) is the source of troubles in the $\tilde{\omega} \rightarrow \infty$ limit of BD theory; since $\nabla_\mu \sigma \approx \mp (3/2\tilde{\omega})^{1/2} \nabla_\mu \ln \phi$, the second term in the right hand side of Eq. (2.2) does not go to zero in the $\tilde{\omega} \rightarrow \infty$ limit, and Eq. (2.2) does not reduce to the Einstein equation with the same $T_{\mu\nu}$.

When $T \neq 0$, conformal invariance is broken, and the conformal transformation approach cannot be applied. Instead, one has to resort to the order of magnitude estimate of Ref. [10] to derive Eq. (1.1) instead of (1.2). In the $T \neq 0$ case, we still lack a rigorous mathematical derivation of Eq. (1.1).

4 Conclusions

Only recently it was realized [16] that the source of troubles in obtaining GR as the $\omega \rightarrow \infty$ limit of BD theory is related to the vanishing of the trace T of the matter energy–momentum tensor. The approach based on conformal transformations allows one to understand the precise relation between the $\omega \rightarrow \infty$ limit and the vanishing of T. The failure to obtain the correct GR limit when $T = 0$ is explained in terms of the invariance of the theory (when $T = 0$) under the group of conformal transformations $\mathcal{F}_\alpha$

given by Eqs. (2.4), (2.7) and (2.8). Since the $\omega \to \tilde{\omega}$ parameter change (including the case $\tilde{\omega} = \infty$) simply moves BD theory within the equivalence class $\mathcal{E}$, and GR does not belong to $\mathcal{E}$, the attempts to obtain GR as the $\omega \to \infty$ limit of BD theory are doomed to failure. It is only when matter with $T \neq 0$ is included into the BD action that this is possible, due to the breaking of conformal invariance and to the fact that the change $\omega \to \tilde{\omega}$ no longer moves the theory within a restricted equivalence class.

The asymptotic behaviour of the BD scalar in the $\omega \to \infty$ limit, under the condition $T = 0$, is given by Eq. (1.2), which receives a sound mathematical justification for the first time in the conformal transformation approach. The latter is only applicable in vacuum ($T_{\mu\nu} = 0$), or in the presence of matter satisfying the condition $T = 0$.

Finally, we point out a issue of potential interest: consider the differential equation

$$L(a) f(x^{\alpha}) = 0 \ , \tag{4.1}$$

where $L(a)$ is a partial differential operator depending on a parameter a. Let L_0 be the limit of $L(a)$ as $a \to 0$, and let f_0 be the limit of a solution $f(x^{\alpha})$ of Eq. (4.1) as $a \to 0$. If ψ is a solution of the equation $L_0 f = 0$, then in general one has $\psi \neq f_0$. Although the $\omega \to \infty$ limit of the BD field equations reproduces the Einstein equations when $T \neq 0$, it is not trivial that a BD exact solution tends to the corresponding GR solution in the same limit. To the best of our knowledge, this property of the BD field equations has not been investigated in the literature, and also within the context of GR a spacetime may not have a well–defined limit as some parameter varies [20]. This issue will be investigated in the future.

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